Indian Statistical Institute, Bangalore

B. Math.

First Year, First Semester

Analysis I

Final Examination — Date : Nov. 11, 2019

Total Marks: 105. Maximum marks: 100. — Time: 3 hours

Teacher: B V Rajarama Bhat

Notation: $\mathbb{R}$ - is the set of real numbers.

(1) Let M be the set of polynomials with rational coefficients. Determine as to whether M is countable or uncountable. Prove your claim. [15]

(2) Suppose $\{x_n\}_{n\geq 1}$ is a bounded sequence of real numbers. Show that it is convergent if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. [15]

(3) Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of positive real numbers. For $n \geq 1$, take $b_n = (-1)^{n+1}a_n$. Show that the series $\sum_{n=1}^{\infty} b_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$. [15]

(4) (i) Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function. Show that f attains its supremum and infimum.

(ii) Given an example of a bounded continuous function $m : \mathbb{R} \to \mathbb{R}$ which does not attain its supremum and infimum. [15]

(5) (i) Let $g : (0, 1] \to \mathbb{R}$ be a continuous function. A function h on $[0, 1]$ is said to be an extension of g if $h(x) = g(x)$ for all $x \in (0, 1]$. Show that g can be extended to a continuous function on $[0, 1]$ if and only if g is uniformly continuous.

(ii) For a real number y, let $\{y\} := y - [y]$ denote the fractional part of y. Determine as to $v : (0, 1] \to \mathbb{R}$ defined by

$$v(x) = \frac{1}{1 + (1 - \{\frac{1}{x}\})^2},\ x \in (0, 1]$$

is uniformly continuous. [15]

(6) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(x) = |(x - 2)^2(x - 1)|$$

for all $x \in \mathbb{R}$. Find the points where h is differentiable and where it is second differentiable. Prove your claim. [15]

(7) State and prove Taylor's theorem for a real valued function on an interval $[a, b]$. [15]